JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **213,** 340–353 (1997)
ARTICLE NO. AY975542

Lévy–Khintchine Formulas Independent of a Lévy Function

Dragu Atanasiu

Department of Mathematics, Chalmers University of Technology and University of

Submitted by N. H. Bingham

Received June 13, 1996

In this paper we obtain integral representations, independent of a Lévy function, for negative definite functions with real part bounded below defined on a commutative involutive semigroup and for continuous negative definite functions defined on the group $\mathbb{R}^n$. © 1997 Academic Press

1. INTRODUCTION

The integral representations of negative definite functions, known as Lévy–Khintchine formulas, depend on a Lévy function (see [3, p. 108, Theorem 3.19; 6, p. 316, Theorem 8]).

The existence of a Lévy function is proved in [7] for locally compact groups and in [4] for commutative involutive semigroups with a neutral element.

In Section 3 of this paper we obtain integral representations for negative definite functions, defined on a commutative involutive semigroup, such that the proof and the result are independent of a Lévy function.

These integral representations are given in Theorem 2 of this paper and generalize [2, p. 239, Theorem 2].

To obtain them, we use a theorem which is related to a Choquet type result on adapted spaces (see [2, p. 238, Theorem 1]) and is proved in Section 2.

We also give, using the proof of Theorem 2, Lévy–Khintchine formulas independent of a Lévy function for continuous negative definite functions defined on $\mathbb{R}^n$ (see [5] for the classical Lévy–Khintchine formula).

0022-247X/97 $25.00

2. A REPRESENTATION THEOREM

Let X be a locally compact Hausdorff space. We denote by $\mathbf{C}(X)$ the set $\{f\colon X \to \mathbb{R} \mid f$ continuous and with compact support$\}$ and by $\mathbf{C}_+(X)$ the set $\{f \in \mathbf{C}(X) \mid f \geq 0\}$.

THEOREM 1. *Let V be a linear space of real continuous functions on X, such that $V \supset \mathbf{C}(X)$, and $L\colon V \to \mathbb{R}$ a linear functional, such that $L(f) \geq 0$ for every $f \in V_+$, where $V_+ = \{f \in V \mid f \geq 0\}$. The restriction of L to $\mathbf{C}(X)$ is represented by a positive Radon measure μ with the following properties*:

(i) *every function of V_+ is μ-integrable and we have $L(f) \geq \int_X f(x)\, d\mu(x)$ for $f \in V_+$*;

(ii) *a function $f \in V$ is μ-integrable and satisfies*

$$L(f) = \int_X f(x)\, d\mu(x)$$

if the following condition is satisfied: *for every real number $\varepsilon > 0$ there exists a function $h \in V_+$ with $L(h) \leq \varepsilon$ and a compact $K \subset X$ such that*

$$|f(x)| \leq h(x) \qquad \textit{for } x \in X - K.$$

Proof. If $f \in V_+$, $g \in \mathbf{C}_+(X)$, and $f \geq g$, then

$$L(f) \geq L(g) = \mu(g).$$

It results that f is μ-integrable and $L(f) \geq \mu(f)$, which proves (i).

Let f be a function of V which satisfies the condition from (ii).

Take $\varepsilon > 0$. Choose a function $h \in V_+$ with $L(h) < \varepsilon$ and a compact set $K \subset X$ such that

$$|f(x)| \leq h(x), \qquad x \in X \setminus K.$$

This implies that f is μ-integrable and consequently we can find a compact $K' \subset X$ such that $\int_{X \setminus K'} |f|\, d\mu \leq \varepsilon$.

We choose a continuous function $\varphi\colon X \to [0, 1]$ with compact support such that $\varphi(x) = 1$ for $x \in K \cup K'$. We have

$$-h \leq f - f\varphi \leq h.$$

The positivity of L yields

$$\left| L(f) - \int f\varphi\, d\mu \right| \leq \varepsilon.$$

We obtain

$$\left|L(f) - \int f\,d\mu\right| \leq \left|L(f) + \int f\varphi\,d\mu\right| + \left|\int f\varphi\,d\mu - \int f\,d\mu\right| \leq 2\varepsilon,$$

which finishes the proof. ∎

3. INTEGRAL REPRESENTATIONS FOR NEGATIVE DEFINITE FUNCTIONS ON AN INVOLUTIVE SEMIGROUP

Let $(S, +, *)$ be a commutative involutive semigroup with neutral element 0 (see [3, p. 86]). We say that a function $\varphi\colon S \to \mathbb{C}$ is positive definite on S if for each natural number $n \geq 1$, each family $c_1, \dots, c_n$ of complex numbers and each family $x_1, \dots, x_n$ of elements of S, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$

A function $\varphi\colon S \to \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that a hermitian function $\varphi\colon S \to \mathbb{C}$ is negative definite on S if for each natural number $n \geq 2$, each family $c_1, \dots, c_n$ of complex numbers such that $c_1 + \cdots + c_n = 0$ and each family $x_1, \dots, x_n$ of elements of S, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$

A function $\rho\colon S \to \mathbb{C}$ is called a character if

(i) $\rho(0) = 1$;
(ii) $\rho(x + y) = \rho(x)\rho(y)$ for $x, y \in S$;
(iii) $\rho(x^*) = \overline{\rho(x)}$ for $x \in S$.

We denote by $\hat{S}$ the set of all bounded characters of S, by which we mean

$$\hat{S} = \{\rho\colon S \to \mathbb{C} \mid \rho \text{ is a character of } S;\ |\rho(x)| \leq 1,\ x \in S\}.$$

Let A be a subset of S such that $0 \in A$. We define $\Lambda := \{\rho \in \hat{S} \mid \rho(A) \subset \mathbb{R}_+\}$ and $\Omega := \{\rho \in \Lambda \mid \rho \not\equiv 1\}$.

With the product topology, $\hat{S}$ and Λ are compact spaces and Ω is a locally compact space.

THEOREM 2. *For a function* $\varphi\colon S \to \mathbb{C}$ *the following conditions are equivalent*:

(i) *the functions* $(x \mapsto \varphi(x+a))_{a \in A}$ *are negative definite on* S *and* φ *has real part bounded below*;

(ii) *there are a real number* C, *a negative definite function* $q\colon S \to [0,\infty[$ *such that*

$$q(x) + q(y) = \tfrac{1}{2}(q(x+y) + q(x^* + y)), \qquad x, y \in S,$$

and a positive Radon measure μ *on* Ω *with the functions* $(\rho \mapsto (1 - \operatorname{Re}\rho(x)))_{x \in S}$ μ*-integrable, which satisfy*

$$\operatorname{Re}\varphi(x) = C + q(x) + \int_\Omega (1 - \operatorname{Re}\rho(x))\, d\mu(\rho), \qquad x \in S$$

and

$$-\operatorname{Im}\varphi(x+y) + \operatorname{Im}\varphi(x) + \operatorname{Im}\varphi(y)$$
$$= \int_\Omega (\operatorname{Im}\rho(x+y) - \operatorname{Im}\rho(x) - \operatorname{Im}\rho(y))\, d\mu(\rho), \qquad x, y \in S.$$

C, q, *and* μ *are uniquely determined by* φ.

Proof. (i) $\Rightarrow$ (ii). Using [3, p. 74, Theorem 2.2], it results that for every $a \in A$ and every $t \in]0,\infty[$ the function $\psi_{t,a}\colon S \to \mathbb{C}$ defined by $\psi_{t,a}(x) = e^{-t\varphi(x+a)}$ is positive definite. Consequently, using also the fact that for every $t \in]0,\infty[$ the function $\psi_{t,0}$ is bounded, we find as in [8, p. 900, Proposition 1] or [1, p. 96, Theorem 2.5] positive Radon measures $(\mu_t)_{t \in]0,\infty[}$ on Λ such that

$$e^{-t\varphi(x)} = \int_\Lambda \rho(x)\, d\mu_t(\rho), \qquad x \in S.$$

We denote by V the set

$$\Big\{ f\colon \Omega \to \mathbb{R} \,|\, f = F|_\Omega,\ F\colon \Lambda \to \mathbb{R},\ F \text{ continuous},$$
$$F(\theta) = 0,\ \lim_{t \to 0} \frac{1}{t} \int_\Lambda F(\rho)\, d\mu_t(\rho) \text{ exist in } \mathbb{R} \Big\},$$

where $f = F|_\Omega$ means that f is the restriction of F to Ω and $\theta\colon S \to \mathbb{C}$ is defined by $\theta(x) = 1$ for every $x \in S$.

V is a real vector space of continuous functions and the function $L\colon V \to \mathbb{R}$ defined by $L(f) = \lim_{t \to 0} (1/t) \int_\Lambda F(\rho)\, d\mu_t(\rho)$ is a linear functional on V such that $L(f) \geq 0$ for $f \in V_+$.

Let $\mathcal{F}$ denote the set of all families $(a_x)_{x \in S}$ of complex numbers such that $a_x \neq 0$ only for a finite number of x and which satisfy the relation $\sum_{x \in S} a_x = 0$.

Let U denote the set

$$\left\{ f\colon \Omega \to \mathbb{R} \mid f(\rho) = \sum_{x \in S} a_x \rho(x), (a_x)_{x \in S} \in \mathcal{F} \right\}.$$

Equivalently, $f \in U$ iff $f = F\mid_{\Omega}$ where

$$F \in \operatorname{span}\big(\{\rho \mapsto \rho(x) \mid x \in S\}\big) \cap \mathbb{R}^{\Lambda} \qquad \text{and} \qquad F(\theta) = 0.$$

U is a real vector space. We shall prove that U is a subspace of V.

If we take $(a_x)_{x \in S} \in \mathcal{F}$ such that the function defined on Ω by

$$\rho \mapsto \sum_{x \in S} a_x \rho(x)$$

is in U, we have

$$\sum_{x \in S} a_x \left(\frac{e^{-t\varphi(x)} - 1}{t} \right) = \frac{1}{t} \int_{\Lambda} \left(\sum_{x \in S} a_x \rho(x) \right) d\mu_t(\rho).$$

Letting t tend to 0 we obtain that the function $(\rho \mapsto \sum_{x \in S} a_x \rho(x))$ is in V and that

$$L\left(\rho \mapsto \sum_{x \in S} a_x \rho(x) \right) = - \sum_{x \in S} a_x \varphi(x).$$

Next we prove that $\mathcal{C}(\Omega) \subset V$.

Let $f \in \mathcal{C}(\Omega)$, $f \not\equiv 0$, and let F be its zero extension to Λ.

We have $\operatorname{supp} f \subset \Omega = \bigcup_{x \in S} \{\rho \in \Omega \mid |1 - \rho(x)| > 0\}$ and consequently we can find a natural number $n \geq 1$ and $a_1, \ldots, a_n$ elements of S such that $\sum_{j=1}^{n} |1 - \rho(a_j)| > 0$ on $\operatorname{supp} f$.

For every $x \in S$ the functions defined on Ω by $\rho \mapsto \operatorname{Re} \rho(x) \sum_{j=1}^{n} |1 - \rho(a_j)|^2$ and $\rho \mapsto \operatorname{Im} \rho(x) \sum_{j=1}^{n} |1 - \rho(a_j)|^2$ belong to U. It results from the inclusion $U \subset V$ that

$$\lim_{t \to 0} \frac{1}{t} \int_{\Lambda} \rho(x) \left(\sum_{j=1}^{n} |1 - \rho(a_j)|^2 \right) d\mu_t(\rho)$$

exists in $\mathbb{C}$. We denote by $u(x)$ this limit.

The function $u\colon S \to \mathbb{C}$ is bounded and the functions $(x \mapsto u(x + a))_{a \in A}$ are positive definite. Using [8, p. 900, Proposition 1] or [1, p. 96, Theorem

2.5], we obtain a positive Radon measure ν on Λ such that

$$u(x) = \int_\Lambda \rho(x)\, d\nu(\rho), \qquad x \in S.$$

We have, by a slight modification of Theorem 2.11 from [3, p. 97] that

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}\int_\Lambda F(\rho)\, d\mu_t(\rho) &= \lim_{t\to 0} \frac{1}{t}\int_\Lambda G(\rho)\left(\sum_{j=1}^n |1-\rho(a_j)|^2\right) d\mu_t(\rho) \\ &= \int_\Lambda G(\rho)\, d\nu(\rho) \end{aligned}$$

(where $G\colon \Lambda \to \mathbb{C}$ is the continuous function defined by $G(\rho) = f(\rho)/\sum_{j=1}^n |1-\rho(a_j)|^2$ for $\rho \in \operatorname{supp} f$ and $G(\rho) = 0$ for $\rho \in \Lambda \setminus \operatorname{supp} f$) which means that $f \in V$.

Let x, y be elements of S and ε a real number such that $0 < \varepsilon < 1$. Let $K_{\varepsilon,y}$ (resp. $K'_{\varepsilon,y}$) be the compact $\{\rho \in \Omega \mid \operatorname{Re}\rho(y) \le 1-\varepsilon\}$ (resp. $\{\rho \in \Omega \mid |\operatorname{Im}\rho(y)| \ge \varepsilon\}$).

We have

$$(1-\operatorname{Re}\rho(x))(1-\operatorname{Re}\rho(y)) \le \varepsilon(1-\operatorname{Re}\rho(x))$$

for $\rho \in \Omega - K_{\varepsilon,y}$ and

$$|(1-\operatorname{Re}\rho(x))\operatorname{Im}\rho(y)| \le \varepsilon(1-\operatorname{Re}\rho(x))$$

for $\rho \in \Omega - K'_{\varepsilon,y}$.

Theorem 1 yields a positive Radon measure μ on Ω such that the elements of V_+ are μ-integrable and we have

$$-\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x^*) \ge \int_\Omega (1-\operatorname{Re}\rho(x))\, d\mu(\rho), \tag{1}$$

$$\begin{aligned} -\varphi(0) &+ \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x^*) + \frac{1}{2}\varphi(y) + \frac{1}{2}\varphi(y^*) \\ &- \frac{1}{4}(\varphi(x+y) + \varphi(x^*+y) + \varphi(x+y^*) + \varphi(x^*+y^*)) \\ &= \int_\Omega (1-\operatorname{Re}\rho(x))(1-\operatorname{Re}\rho(y))\, d\mu(\rho), \end{aligned} \tag{2}$$

$$
\begin{aligned}
&-\frac{1}{2i}(\varphi(y)-\varphi(y^*))\\
&\quad+\frac{1}{4i}(\varphi(x+y)+\varphi(x^*+y)-\varphi(x+y^*)-\varphi(x^*+y^*))\\
&\quad=\int_\Omega(1-\operatorname{Re}\rho(x))\operatorname{Im}\rho(y)\,d\mu(\rho),
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&-\frac{1}{2i}(\varphi(x)-\varphi(x^*))\\
&\quad+\frac{1}{4i}(\varphi(x+y)+\varphi(y^*+x)-\varphi(y+x^*)-\varphi(y^*+x^*))\\
&\quad=\int_\Omega(1-\operatorname{Re}\rho(y))\operatorname{Im}\rho(x)\,d\mu(\rho).
\end{aligned}
\tag{4}
$$

If we denote by $q\colon S\to\mathbb{R}$ the function defined by $q(x)=-\varphi(0)+\operatorname{Re}\varphi(x)-\int_\Omega(1-\operatorname{Re}\rho(x))\,d\mu(\rho)$, then the relation (1) gives $q(x)\geq 0$, $x\in S$.

Choosing a natural number $n\geq 2$, $c_1,\dots,c_n$ complex numbers, such that $c_1+\cdots+c_n=0$, and $x_1,\dots,x_n$ elements of S, we have

$$
\begin{aligned}
&\sum_{j,k=1}^n c_j\bar{c}_k q(x_j+x_k^*)\\
&\qquad=\sum_{j,k=1}^n c_j\bar{c}_k\varphi(x_j+x_k^*)+\int_\Omega\left|\sum_{j=1}^n c_j\rho(x_j)\right|^2 d\mu(\rho).
\end{aligned}
$$

This proves that q is negative definite because the function defined on Ω by

$$
\rho\mapsto\left|\sum_{j=1}^n c_j\rho(x_j)\right|^2\qquad\text{is in } V_+.
$$

The formula (2) gives

$$
q(x)+q(y)=\tfrac{1}{2}(q(x+y)+q(x^*+y)).
$$

From (3) and (4) we obtain

$$
-\frac{1}{2i}(\varphi(y)-\varphi(y^*))-\frac{1}{2i}(\varphi(x)-\varphi(x^*))
$$

$$+\frac{1}{2i}(\varphi(x+y)-\varphi(x^*+y^*))$$

$$=\int_\Omega (1-\operatorname{Re}\rho(x))\operatorname{Im}\rho(y)\,d\mu(\rho)$$

$$+\int_\Omega (1-\operatorname{Re}\rho(y))\operatorname{Im}\rho(x)\,d\mu(\rho)$$

$$=\int_\Omega (\operatorname{Im}\rho(x)+\operatorname{Im}\rho(y)-\operatorname{Im}\rho(x+y))\,d\mu(\rho)$$

which gives the second integral formula from the theorem.

(ii) ⇒ (i). Let n be a natural number ≥ 2, $c_1,\dots,c_n$ complex numbers, such that $c_1+\cdots+c_n=0$, and $x_1,\dots,x_n$ elements of S. If we have the integral representations of (ii) and if $a\in A$, it follows that

$$\sum_{j,k=1}^n c_j\bar{c}_k\varphi(x_j+x_k^*+a)$$

$$=\sum_{j,k=1}^n c_j\bar{c}_k\operatorname{Re}\varphi(x_j+x_k^*+a)+\sum_{j,k=1}^n c_j\bar{c}_k\operatorname{Im}\varphi(x_j+x_k^*+a)$$

$$=\sum_{j,k=1}^n c_j\bar{c}_k q(x_j+x_k^*+a)$$

$$+\sum_{j,k=1}^n c_j\bar{c}_k\big(\operatorname{Im}\varphi(x_j+a)+\operatorname{Im}\varphi(x_k^*)\big)$$

$$+\int_\Omega\Bigg(-\left|\sum_{j=1}^n c_j\rho(x_j)\right|^2\rho(a)$$

$$+\sum_{j,k=1}^n c_j\bar{c}_k\big(\operatorname{Im}\rho(x_j+a)+\operatorname{Im}\rho(x_k^*)\big)\Bigg)\,d\mu(\rho)$$

$$=\sum_{j,k=1}^n c_j\bar{c}_k q(x_j+x_k^*+a)-\int_\Omega\left|\sum_{j=1}^n c_j\rho(x_j)\right|^2\rho(a)\,d\mu(\rho).$$

Now using the properties of q, we obtain

$$\sum_{j,k=1}^n c_j\bar{c}_k q(x_j+x_k^*+a)$$

$$=\frac{1}{2}\sum_{j,k=1}^n c_j\bar{c}_k\big(q(x_j+x_k^*+a)-2q(x_j+a)-2q(x_k)\big)$$

$$+\frac{1}{2}\sum_{j,k=1}^{n} c_j\bar{c}_k\big(q(x_j+x_k^*+a)-2q(x_j+x_k^*)-2q(a)\big)$$
$$+\sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*)$$
$$=\frac{1}{2}\sum_{j,k=1}^{n} c_j\bar{c}_k\big(q(x_j+x_k+a)+q\big(x_j+x_k^*+a^*\big)\big)$$
$$+\sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*)$$
$$=\sum_{j,k=1}^{n} c_j\bar{c}_k\big(q(x_j)+q(x_k+a)\big)+\sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*)$$
$$=\sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*)\le 0.$$

We have shown that the function $x \mapsto q(x+a)$ is negative definite and consequently that the function $x \mapsto \varphi(x+a)$ is negative definite for every $a \in A$. This proves the implication (ii) $\Rightarrow$ (i) because it is clear that φ has the real part bounded below.

The unicity of μ results from the equality

$$-\varphi(x)+\varphi(x+y)+\varphi(x+y^*)$$
$$-\frac{1}{4}\big(\varphi(x+2y)+2\varphi(x+y+y^*)+\varphi(x+2y^*)\big)$$
$$=\int_\Omega \rho(x)(1-\operatorname{Re}\rho(y))^2\,d\mu(\rho),\qquad x,y\in S.$$

Unicity of q is a consequence of the unicity of μ because $C=\varphi(0)$. This finishes the proof of the theorem. ∎

Remark 1. In [3, p. 101, Theorem 3.9] it is proved that a function q: $S \to [0,\infty[$, such that

$$q(x)+q(y)=\tfrac{1}{2}\big(q(x+y)+q(x+y^*)\big),\qquad x,y\in S$$

is automatically negative definite.

COROLLARY 1. *We have $\mu = 0$ if and only if $\varphi(x) = C + q(x) + il(x)$, $x \in S$, where C and q are as in the Theorem 2 and l: $S \to \mathbb{R}$ is a function such that $l(x+y) = l(x) + l(y)$, $x, y \in S$, and $l(x^*) = -l(x)$, $x \in S$.*

COROLLARY 2. *We have* $\lim_{t\to 0} \mu_t|_\Omega = \mu$ *vaguely*.

Corollaries 1 and 2 result from the proof of the theorem.

In the following the group $\mathbb{R}^n$ is considered as a $*$-semigroup with the involution $x^* := -x$.

THEOREM 3. *For a function* $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ *the following conditions are equivalent*:

(i) φ *is continuous and negative definite on* $\mathbb{R}^n$;

(ii) *there is a real number* C, *a positive quadratic form* $q\colon \mathbb{R}^n \to \mathbb{R}$ *such that* $q(x) = \sum_{j,k=1}^n a_{jk} x_j x_k$ *with* $a_{jk} \in \mathbb{R}$ *and* $a_{jk} = a_{kj}$, *and a positive Radon measure* μ *on* $\mathbb{R}^n \setminus (0,\dots,0) = \Omega$ *with the function* $g\colon \Omega \to \mathbb{R}$ *defined by* $g(\rho) = \|\rho\|^2/(1+\|\rho\|^2)$ μ*-integrable, which satisfy*

$$\operatorname{Re}\varphi(x) = C + q(x) + \int_\Omega (1 - \cos\langle \rho, x\rangle)\, d\mu(\rho)$$

and

$$\begin{aligned}-\operatorname{Im}\varphi(x+y) &+ \operatorname{Im}\varphi(x) + \operatorname{Im}\varphi(y)\\ &= \int_\Omega (\sin\langle \rho, x+y\rangle - \sin\langle \rho, x\rangle - \sin\langle \rho, y\rangle)\, d\mu(\rho).\end{aligned}$$

C, q, *and* μ *are uniquely determined by* φ.

Proof. By Bochner's theorem the measures μ_t used in the proof of Theorem 2 to represent $e^{-t\varphi}$ are concentrated on $\mathbb{R}^n$.

Let α be a real number >0. Let $h_\alpha\colon \mathbb{R}^n \to [0,\infty[$ be the function defined by $h_\alpha(\rho) = (1/\alpha^n)\int_{[0,\alpha]^n} (1-\cos\langle \rho, x\rangle)\, dx$ (where dx is the Lebesgue measure on $\mathbb{R}^n$).

We have, with Fubini's theorem,

$$\begin{aligned}L(h_\alpha|_\Omega) &= \lim_{t\to 0} \frac{1}{t}\int_{\mathbb{R}^n} h_\alpha(\rho)\, d\mu_t(\rho)\\ &= \frac{1}{\alpha^n}\int_{[0,\alpha]^n} (-\varphi(0) + \operatorname{Re}\varphi(x))\, dx.\end{aligned}$$

It is immediate that for every $\varepsilon > 0$, there is a real number $\alpha > 0$ such that $L(h_\alpha|_\Omega) < \varepsilon$ and that for every $\alpha > 0$ there exists $M > 0$ such that $h_\alpha(\rho) \geq 1/2$ for $\|\rho\| \geq M$.

Let x, y be elements of $\mathbb{R}^n$ and $\varepsilon > 0$. Choose $\alpha > 0$ such that $L(h_\alpha|_\Omega) \leq \varepsilon$. There exists a compact $K_{\varepsilon,y}$ (resp. $K'_{\varepsilon,y}$) of Ω such that

$$\begin{aligned}(1 - \cos\langle \rho, x\rangle)&(1-\cos\langle \rho, y\rangle)\\ &\leq \varepsilon(1-\cos\langle \rho, x\rangle) + 8h_\alpha(\rho) \qquad \text{for } \rho \in \Omega\setminus K_{\varepsilon,y}\end{aligned}$$

(resp. $|(1-\cos\langle\rho,x\rangle)\sin\langle\rho,y\rangle| \leq \varepsilon(1-\cos\langle\rho,x\rangle)+4h_\alpha(\rho)$ for $\rho\in\Omega\setminus K'_{\varepsilon,y}$).

Using these inequalities and Theorem 1, we can obtain the integral representations of (ii) as in the proof of Theorem 2.

The implication (ii) $\Rightarrow$ (i) and the unicity of C, q, and μ also result as in Theorem 2.

To finish the proof, we only have to prove that if a positive Radon measure μ on Ω is such that the functions

$$(\rho\mapsto 1-\cos\langle\rho,x\rangle)_{x\in\mathbb{R}^n}$$

are μ-integrable, then the function $\rho\mapsto\|\rho\|^2/(1+\|p\|^2)$ is also μ-integrable.

Using (1) and Fubini's theorem, we have

$$\begin{aligned}\int_{[0,1]^n}&\left(-\varphi(0)+\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(-x)\right)dx\\&\geq\int_{[0,1]^n}\left(\int_\Omega(1-\cos\langle\rho,x\rangle)\,d\mu(\rho)\right)dx\\&=\int_\Omega\left(1-\int_{[0,1]^n}\cos\langle\rho,x\rangle\,dx\right)d\mu(\rho).\end{aligned}$$

Now it is obvious that the function $\rho\to\|\rho\|^2/(1+\|\rho\|^2)$ is μ-integrable. ∎

In the following $\mathbb{N}$ is the set $\{0,1,2,\dots\}$.

EXAMPLE 1. *Consider the semigroup* $(\mathbb{N}^2,+)$ *with the involution* $(m,n)^*=(n,m)$. *For a function* $\varphi\colon\mathbb{N}^2\to\mathbb{C}$ *the following conditions are equivalent*:

(i) φ *is negative definite and has the real part bounded below*;

(ii) *there are real numbers* C,α,β *such that* $\alpha,\beta\geq 0$ *and a positive Radon measure* μ *on* $\Omega=\{z\in\mathbb{C}\,|\,|z|\leq 1, z\neq 1\}$ *with the function* $x+iy\mapsto 1-x$ μ*-integrable, which satisfy*

$$\operatorname{Re}\varphi(m,n)=C+(m+n)\alpha+(m-n)^2\beta+\int_\Omega(1-\operatorname{Re}z^m\bar{z}^n)\,d\mu(z)$$

and

$$\begin{aligned}\operatorname{Im}(-\varphi(m+p,n+q)&+\varphi(m,n)+\varphi(p,q))\\&=\int_\Omega\operatorname{Im}(z^{m+p}\bar{z}^{n+q}-z^m\bar{z}^n-z^p\bar{z}^q)\,d\mu(z).\end{aligned}$$

C, α, β, and μ are uniquely determined by φ and we have

$$\alpha = -\varphi(0,0) + \frac{1}{2}(\varphi(1,0) + \varphi(0,1)) - \frac{1}{8}(\varphi(2,0) - 2\varphi(1,1) + \varphi(0,2)) - \int_\Omega \left(1 - x - \frac{1}{2}y^2\right) d\mu(x+iy)$$

and

$$\beta = \frac{1}{8}(\varphi(2,0) + 2\varphi(1,1) + \varphi(0,2)) - \frac{1}{2}\int_\Omega y^2\, d\mu(x+iy).$$

Proof. We denote by Λ the set $\{z \in \mathbb{C} \mid |z| \le 1\}$. We have

$$\hat{\mathbb{N}}^2 = \big\{\rho\colon \mathbb{N}^2 \to \mathbb{C} \mid \rho(0,0) = 1;\ \rho(m,n) = \overline{\rho(n,m)};\ \rho(m+p, n+q) = \rho(m,n)\cdot\rho(p,q);\ |\rho(m,n)| \le 1, m,n,p,q \in \mathbb{N}\big\}.$$

Let $z \in \Lambda$. The function $\rho_z\colon \mathbb{N}^2 \to \mathbb{C}$ given by $\rho_z(m,n) = z^m\bar{z}^n$ is in $\hat{\mathbb{N}}^2$ and the mapping $z \mapsto \rho_z$ is a topological isomorphism of Λ onto $\hat{\mathbb{N}}^2$. Using this isomorphism, Example 1 is a particular case of Theorem 2 and we only have to calculate $q(m,n)$ where

$$q(m,n) = -\varphi(0,0) + \operatorname{Re}\varphi(m,n) - \int_\Omega (1 - \operatorname{Re} z^m\bar{z}^n)\, d\mu(z).$$

As in the proof of Theorem 2, we notice that the function defined on Ω by

$$x + iy \mapsto (1-x)^m y^n$$

is μ-integrable for $m \ge 1$ or $n \ge 2$ and we have

$$L\big(x+iy \mapsto (1-x)^m y^n\big) = \int_\Omega (1-x)^m y^n\, d\mu(x+iy)$$

for $m \ge 2$ or $n \ge 3$ or ($m \ge 1$ and $n \ge 1$).

Using this and the binomial theorem, we obtain that

$$\begin{aligned} L\bigg(&x+iy \mapsto 1 - \operatorname{Re}(x+iy)^m(x-iy)^n \\ &- (m+n)\left(1 - x - \frac{1}{2}y^2\right) - \frac{1}{2}(m-n)^2y^2\bigg) \\ &= \int_\Omega \bigg(1 - \operatorname{Re}(x+iy)^m(x-iy)^n \\ &\qquad - (m+n)\left(1 - x - \frac{1}{2}y^2\right) - \frac{1}{2}(m-n)^2y^2\bigg)\, d\mu(x+iy). \end{aligned}$$

This is equivalent to $q(m,n) = (m+n)\alpha + (m-n)^2\beta$, where $\alpha = L(x + iy \mapsto 1 - x - \frac{1}{2}y^2) - \int_\Omega (1 - x - \frac{1}{2}y^2)\, d\mu(x+iy)$ and

$$\beta = L\left(x + iy \mapsto \frac{1}{2}y^2\right) - \frac{1}{2}\int_\Omega y^2\, d\mu(x+iy).$$

We have $1 - x - \frac{1}{2}y^2 = \frac{1}{2}(1-x)^2 + \frac{1}{2}(1 - x^2 - y^2) \geq 0$, which implies that $\alpha \geq 0$. That $\beta \geq 0$ is obvious. ∎

Remark 2. The integral representation of negative definite functions considered in Example 1, which uses a Lévy function, is in [3, p. 119, Proposition 4.15].

EXAMPLE 2. *Consider the semigroup* $(\mathbb{N}^2, +, *)$ *as in Example* 1. *For a function* $\varphi\colon \mathbb{N}^2 \to \mathbb{C}$ *the following conditions are equivalent*:

(i) φ *and the function* $(m,n) \mapsto \varphi(m+1,n)$ *are negative definite and* φ *has the real part bounded below*;

(ii) *there are real numbers* C, α, β, γ *such that* $\alpha, \beta \geq 0$ *and a positive Radon measure* μ *on* $\Omega = [0,1[$ *with the function* $x \mapsto 1 - x$ μ*-integrable, which satisfy*

$$\varphi(m,n) = C + (m+n)\alpha + (m-n)^2\beta + i(m-n)\gamma + \int_\Omega (1 - x^{m+n})\, d\mu(x).$$

C, α, β, γ, *and* μ *are uniquely determined by* φ.

The proof is similar to the proof of Example 1.

ACKNOWLEDGMENT

I am indebted to the referee for valuable comments and suggestions.

REFERENCES

1. D. Atanasiu, Un théorème du type Bochner-Godement et le problème des moments, *J. Funct. Anal.* **92** (1990), 92–102.
2. D. Atanasiu, A Lévy-Khintchine formula for semigroups and related problems: An adapted spaces approach, *J. Math. Anal. Appl.* **198** (1996), 237–247.

3. C. Berg, J. P. R. Christensen, and P. Ressel, ‘‘Harmonic Analysis on Semi-groups,’’ Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1984.
4. H. Buchwalter, Les functions de Lévy existent!, *Math. Ann.* **274** (1986), 31–34.
5. P. Courrège, Générateur infinitesimal d'un semi-groupe de convolution sur $\mathbb{R}^n$ et formule de Lévy-Khintchine, *Bull. Sci. Math.* **88** (1964), 3–30.
6. G. Forst, The Lévy-Hinčin representation of negative definite functions, *Z. Wahrsch. Verw. Gebiete* **34** (1976), 313–318.
7. K. R. Parthasarathy, R. Ranga Rao, and S. R. S. Varadhan, Probability distributions on locally compact abelian groups, *Illinois J. Math.* **7** (1963), 337–369.
8. P. Ressel, De Finetti-type theorems: An analytical approach, *Ann. Probab.* **13** (1985), 898–922.